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# Canonical formulation of shallow water waves†

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**Abstract.** We introduce a new potential and use it with the velocity potential to construct a new variational principle for long surface waves in shallow water. Applying Dirac's theory of constraints, we cast it into canonical form and obtain an explicit expression for the exact Hamiltonian.

## 1. Introduction

The equations governing surface gravity waves on water were given a variational formulation by Luke (1967), and subsequently Zakharov (1968), Broer (1974) and Miles (1977) have cast it into Hamiltonian form. In Luke's Lagrangian only the velocity potential enters as an independent field, while the domain of integration of the action is restricted to yield the boundary conditions at the free surface and the bottom. In the Hamiltonian formulation of this theory the velocity potential at, and the displacement of, the free surface turn out to be canonically conjugate variables. However, it has not been possible to write the Hamiltonian in closed form using these canonical variables and a great deal of work has been done to obtain approximate expressions (cf Miles 1981). It is the purpose of this paper to present an entirely new formulation of the equations of motion for the special case of long waves in shallow water in terms of potentials which finally enables us to construct an explicit expression for the exact Hamiltonian.

We start with the well known formulation of the full set of equations governing long waves in shallow water as a system of conservative equations (Stoker 1957, Whitham 1974). In $1+1$ space and time dimensions these equations can be expressed as the integrability condition for two potentials which for consistency satisfy a pair of coupled, first-order, nonlinear partial differential equations. One of them is the familiar velocity potential subject to Bernoulli's law and the other one appears to be new. The equations of motion for surface waves can be derived from a new variational principle where the action is a functional of these potentials. As Seliger and Whitham (1968) have emphasised, it is crucial to introduce potentials for various phenomenological fields in order to construct variational principles in fluid mechanics. The difference between our approach and that of Seliger and Whitham lies in the fact that we have introduced the potentials directly for the conservative equations themselves rather than using Clebsch velocity potentials. The new variational principle is, once again, an expression of the principle of stationary pressure. The action is now defined over an arbitrary domain and independent variations with respect to the two potentials

† For Professor Ata Nutku on his 80th Birthday.

 

yield all the equations of motion. The Hamiltonian formulation of this theory requires the use of Dirac's theory of constrained systems (Dirac 1964) because the new Lagrangian is degenerate. We find that there are two second class constraints in this problem, and applying Dirac's theory construct the total Hamiltonian in terms of the potentials and their canonically conjugate momenta. Thus we obtain the canonical formulation of surface waves in shallow water.

## 2. Potentials

The theory of shallow water waves is well known. The continuity and Euler equations of hydrodynamics in one space dimension can be written in the form

$$u_t + uu_x + 2cc_x = H_x, \qquad 2c_t + 2uc_x + cu_x = 0, \tag{1a, b}$$

where $u$ and $c$ are the velocities of the fluid and of the disturbance with respect to the fluid respectively. These quantities are functions of $x$ and $t$ while the 'depth' $H$ is a given function which depends only on $x$. Subscripts denote partial derivatives. In (1) we are using a notation derived from gas dynamics because of the analogy between our problem and the flow of a compressible gas in one direction with adiabatic index $\gamma = 2$ (Stoker 1957). See also appendix 1.

The introduction of potentials into this problem rests on the following observation. Let us consider the one-forms

$$\alpha = u\,\mathrm{d}x - (\tfrac{1}{2}u^2 + c^2 - H)\,\mathrm{d}t, \qquad \omega = c^2(\mathrm{d}x - u\,\mathrm{d}t) \tag{2a, b}$$

and note that the conditions for them to be closed,

$$\mathrm{d}\alpha = 0, \qquad \mathrm{d}\omega = 0, \tag{3a, b}$$

are equivalent to equations (1). Then from (3) we have, locally, using Poincaré's lemma,

$$\alpha = \mathrm{d}\Phi, \qquad \omega = \mathrm{d}\Psi, \tag{4a, b}$$

where $\Phi$ and $\Psi$ are 0-forms, i.e. scalars which will be called potentials. From (2) and (4) we obtain

$$\Phi_x = u, \qquad \Phi_t = -\tfrac{1}{2}u^2 - c^2 + H, \tag{5a, b}$$

$$\Psi_x = c^2, \qquad \Psi_t = -uc^2, \tag{5c, d}$$

and we can check that equations (1) are the integrability conditions of these equations. In (5$a$) we find the definition of the velocity potential. From (5) it follows that

$$\Phi_t + \tfrac{1}{2}\Phi_x^2 + \Psi_x = H, \qquad \Psi_t + \Phi_x\Psi_x = 0, \tag{6a, b}$$

which are coupled, first-order, nonlinear partial differential equations satisfied by the potentials. Differentiation of equations (6) with respect to $t$ or $x$ results either in identities or the original equations (1), and this remains true if in addition we introduce arbitrary constants into the right-hand sides of equations (6). Equation (6$a$) is Bernoulli's integral.

Equations (1) and (6) are two systems of nonlinear partial differential equations describing the same phenomenon. In the first case we employ the traditional variables of fluid dynamics while in the latter potentials are used to write the equations of surface waves in shallow water. The connection between the two sets of variables is given by equations (5) for which (1) and (6) are the integrability and compatibility conditions.

## 3. Variational principle

The potentials $\Phi$ and $\Psi$ which we have introduced in (4) can be used to obtain equations (1) for shallow water waves from a variational principle. For this purpose we consider

$$\delta I = 0, \qquad I = \int \mathscr{L}\, \mathrm{d}x\, \mathrm{d}t, \tag{7}$$

where

$$\mathscr{L}_1 = \Phi_t \Psi_x + \Psi_t \Phi_x + \Phi_x^2 \Psi_x + \Psi_x^2 - 2\Psi_x H \tag{8}$$

is the Lagrangian density. From (7) and (8) variation with respect to $\Phi$ and $\Psi$ yields equations (1) upon using (5). $H$ is considered to be a given function which we do not vary. In contrast to the earlier variational principle of Luke, the domain of integration of the action can now be chosen arbitrarily.

We shall now cast this variational principle into Hamiltonian form. The Lagrangian (8) has the interesting property that it depends linearly on the velocities $\Phi_t$, $\Psi_t$. This fact is of crucial importance in passing to a Hamiltonian formulation. Thus we find that the canonical momenta

$$\Pi_\Phi = \Psi_x, \qquad \Pi_\Psi = \Phi_x \tag{9}$$

cannot be inverted for the velocities. Therefore there are constraints and we need to use Dirac's theory to obtain the canonical formalism for this degenerate system. We start with the definition of the constraints

$$C_1 = \Pi_\Phi - \Psi_x \approx 0, \qquad C_2 = \Pi_\Psi - \Phi_x \approx 0, \tag{10$a$, $b$}$$

where $\approx 0$ stands for 'weakly zero'. Using the canonical relations

$$[\Phi(x), \Pi_\Phi(x')] = \delta(x - x'), \qquad [\Psi(x), \Pi_\Psi(x')] = \delta(x - x'), \tag{11}$$

we find that the Poisson brackets of the constraints are given by

$$\begin{aligned} &[C_i(x), C_i(x')] = 0, \qquad i = 1, 2, \text{no sum on } i, \\ &[C_1(x), C_2(x')] = -2\delta_x(x - x'), \end{aligned} \tag{12}$$

with $\delta$ denoting the Dirac delta function. Hence the constraints are second class. The total Hamiltonian of Dirac

$$H_{\mathrm{T}} = \int \mathscr{H}_0\, \mathrm{d}x + \int \mathscr{H}\, \mathrm{d}x \tag{13}$$

consists of a free part

$$\mathscr{H}_0 = \Pi_\Phi \Phi_t + \Pi_\Psi \Psi_t - \mathscr{L}_1 = -\Phi_x^2 \Psi_x - \Psi_x^2 + 2\Psi_x H \tag{14}$$

and a linear combination of constraints

$$\mathscr{H} = \lambda C_1 + \sigma C_2 \tag{15}$$

where $\lambda$, $\sigma$ will be determined from the requirement that the Poisson bracket of the constraints with the total Hamiltonian must vanish. Direct calculation shows that

$$\begin{aligned} &[C_1, H_{\mathrm{T}}] = -2(\Phi_x \Psi_x + \sigma)_x = 0, \\ &[C_2, H_{\mathrm{T}}] = -2(\tfrac{1}{2}\Phi_x^2 + \Psi_x - H + \lambda)_x = 0 \end{aligned} \tag{16}$$

and we shall fix $\lambda$, $\sigma$ by setting the quantities inside the parentheses above equal to zero. In this way we have ignored two arbitrary functions of time which could have been incorporated into $\lambda$ and $\sigma$ corresponding to the possibility of including such functions on the right-hand sides of equations (6). Collecting these results we find the Hamiltonian

$$H_{\mathrm{T}} = \int [\tfrac{1}{2}\Phi_x^2\Psi_x + H\Psi_x - \Pi_\Phi(\tfrac{1}{2}\Phi_x^2 + \Psi_x - H) - \Pi_\Psi\Phi_x\Psi_x]\,\mathrm{d}x. \tag{17}$$

It may be verified that the equations of motion following from this Hamiltonian are equations (6) for the potentials and equations (1) governing the propagation of surface waves in shallow water. From (9) and (5) the associated conservation law reduces to the statement that the one-form

$$\theta = \tfrac{1}{2}c^2(u^2 + c^2 - 2H)\,\mathrm{d}x - uc^2(\tfrac{1}{2}u^2 + c^2 - H)\,\mathrm{d}t \tag{18}$$

is closed. Recently Akyildiz (1982, 1983) has discussed the symplectic structure of shallow water waves in the framework of Manin's formalism (Manin 1978) which is related to the existence of this closed one-form.

## 4. Alternative formulations

The coupled partial differential equations for the potentials can be decoupled by increasing the order of differentiation. The resulting equations can, in turn, be derived from new variational principles which therefore provide alternative formulations of surface waves in shallow water.

For this purpose we note that we can solve for $\Psi_x$ or $\Phi_x$ from ($6a$) or ($6b$) and plug the result into ($6b$) or ($6a$) after differentiating the latter equation with respect to $x$. Thus we find

$$\Phi_{tt} + 2\Phi_x\Phi_{xt} + (\Phi_t + \tfrac{3}{2}\Phi_x^2)\Phi_{xx} = \Phi_{xx}H + \Phi_x H_x, \tag{19}$$

$$\Psi_x^2\Psi_{tt} - 2\Psi_x\Psi_t\Psi_{xt} + (\Psi_t^2 - \Psi_x^3)\Psi_{xx} = -\Psi_x^3 H_x, \tag{20}$$

which are second-order decoupled quasi-linear partial differential equations. Either (19) or (20) is equivalent to equations (6) in that once a solution to (19) or (20) is known the full solution can be constructed by quadratures. These equations can also be obtained from variational principles. We can verify that the action constructed from the Lagrangians

$$\mathscr{L}_2 = \tfrac{1}{2}(\Phi_t + \tfrac{1}{2}\Phi_x^2 - H)^2, \tag{21}$$

$$\mathscr{L}_3 = \tfrac{1}{2}\Psi_x^{-1}\Psi_t^2 - \tfrac{1}{2}\Psi_x^2 + \Psi_x H \tag{22}$$

yields (19) and (20) respectively. Once again, there is no variation with respect to the depth function $H$. The Hamiltonians for $\mathscr{L}_2$ and $\mathscr{L}_3$ can be obtained readily.

## 5. Conclusion

The subject of surface waves in shallow water is a time-honoured one. Since the celebrated work of Riemann in 1859 the method of characteristics has provided us with a very successful approach to this system of equations. Yet it is only recently that variational formulations for surface waves have been given. In this paper a new approach uses two potentials for long waves in shallow water, the integrability conditions for these potentials are the shallow-water equations of motion and for consistency the potentials satisfy a pair of coupled nonlinear partial differential equations. It is necessary to use both these potentials in order to construct a variational principle which requires no restrictions on the domain of integration of the action. This is the important advantage of the new variational principle. The new Lagrangian is, however, degenerate and we had to use Dirac's theory of constraints to cast it into canonical form. The resulting Hamiltonian is expressed very simply in terms of the potentials and their conjugate momenta. It will be most interesting to study the evolution of Cauchy data according to this Hamiltonian; indeed, we may readily expect the new Hamiltonian to play a most important role in understanding the dynamics of surface waves in shallow water with an arbitrarily shaped bottom.

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## Appendix 1. Gas dynamics

The equations of gas dynamics can be formulated using potentials in a fashion similar to the equations of shallow water waves. These equations are

$$D^\varepsilon R^\varepsilon = 0 \tag{A1.1$a$}$$

where $\varepsilon = \pm 1$,

$$D^\varepsilon = \partial/\partial t + (u + \varepsilon c)\partial/\partial x \tag{A1.1$b$}$$

and

$$R^\varepsilon = u + 2\varepsilon c/(\gamma - 1) \tag{A1.1$c$}$$

are the Riemann invariants. As we mentioned earlier, for $\gamma = 2, H = 0$ the two problems coincide. Equations (A1.1) can be recognised as the conditions for the differential one-forms

$$\alpha = u\, \mathrm{d}x - [\tfrac{1}{2}u^2 + (\gamma - 1)^{-1}c^2]\, \mathrm{d}t, \tag{A2.1$a$}$$

$$\omega = c^{2/(\gamma-1)}(\mathrm{d}x - u\, \mathrm{d}t) \tag{A2.1$b$}$$

to be closed. Once again, using Poincaré's lemma, we can introduce the potentials $\Phi$ and $\Psi$ through equations (4). Then we find

$$\Phi_x = u, \qquad \Phi_t = -\tfrac{1}{2}u^2 - (\gamma - 1)^{-1}c^2, \tag{A3.1$a$, $b$}$$

$$\Psi_x = c^{2/(\gamma-1)}, \qquad \Psi_t = -uc^{2/(\gamma-1)} \tag{A1.3$c$, $d$}$$

which results in

$$\Phi_t + \tfrac{1}{2}\Phi_x^2 + (\gamma - 1)^{-1}\Psi_x^{\gamma-1} = 0, \qquad \Psi_t + \Phi_x\Psi_x = 0. \tag{A1.4$a$, $b$}$$

We can verify that (A1.1) and (A1.4) are the integrability and compatibility conditions for (A1.3). From (A1.4) we can obtain decoupled equations for the potentials

$$\Phi_{tt} + 2\Phi_x\Phi_{xt} + [(\gamma-1)\Phi_t + \tfrac{1}{2}(\gamma+1)\Phi_x^2]\Phi_{xx} = 0, \qquad \text{(A1.5)}$$

$$\Psi_x^2\Psi_{tt} - 2\Psi_x\Psi_t\Psi_{xt} + (\Psi_t^2 - \Psi_x^{\gamma+1})\Psi_{xx} = 0, \qquad \text{(A1.6)}$$

along the same lines as before. These equations can be derived from a variational principle with the Lagrangians

$$\mathscr{L}_4 = \Phi_t\Psi_x + \Psi_t\Phi_x + \Phi_x^2\Psi_x + [2/\gamma(\gamma-1)]\Psi_x^\gamma, \qquad \text{(A1.7)}$$

$$\mathscr{L}_5 = [(\gamma-1)/\gamma](\Phi_t + \tfrac{1}{2}\Phi_x^2)^{\gamma/(\gamma-1)}, \qquad \text{(A1.8)}$$

$$\mathscr{L}_6 = \tfrac{1}{2}\Psi_x^{-1}\Psi_t^2 - [\gamma(\gamma-1)]^{-1}\Psi_x^\gamma, \qquad \text{(A1.9)}$$

respectively. The Hamiltonian formulation of $\mathscr{L}_5$ and $\mathscr{L}_6$ is straightforward while in the case of $\mathscr{L}_4$ we can again use Dirac's theory to obtain the Hamiltonian density

$$\mathscr{H}_4 = \tfrac{1}{2}\Phi_x^2\Psi_x + \frac{\gamma-2}{\gamma(\gamma-1)}\Psi_x^\gamma - \Pi_\Phi\left(\tfrac{1}{2}\Phi_x^2 + \frac{1}{\gamma-1}\Psi_x^{\gamma-1}\right) - \Pi_\Psi\Phi_x\Psi_x \qquad \text{(A1.10)}$$

where the definition of momenta is given by (9).

## Appendix 2

We shall now investigate the invariance properties of the equations satisfied by the potentials in order to construct exact solutions. Let us first list these scale transformations. Equations (6) are invariant under the change

$$\Phi \to \alpha\beta\Phi, \qquad \Psi \to \alpha^{3/2}\beta\Psi, \qquad H \to \alpha H, \qquad x \to \alpha^{1/2}\beta x, \qquad t \to \beta t, \qquad \text{(A2.1)}$$

where $\alpha$ and $\beta$ are arbitrary constants. Further, the scale transformations leaving (19) and (20) invariant are given by

$$\Phi \to \Phi, \qquad t \to \alpha^2 t, \qquad x \to \alpha x, \qquad H \to \alpha^{-2}H, \qquad \text{(A2.2)}$$

and

$$\Psi \to \Psi, \qquad t \to \alpha^3 t, \qquad x \to \alpha^2 x, \qquad H \to \alpha^{-2}H \qquad \text{(A2.3)}$$

respectively.

From (A2.1) it is evident that for $H = 0$ there will be a solution of the form $\Phi \sim x^2/t$ and $\Psi \sim x^3/t^2$. This turns out to be the well known simple wave solution. Another solution of (6) follows by considering $\beta = 1$ in (A2.1). In this case we can allow for a quadratically varying depth function and all the space dependences of the potentials are fixed. If we further start with the ansatz

$$\Phi = \tfrac{1}{2}(\dot{w}/w)x^2, \qquad \Psi = x^3/w^3, \qquad H = hx^2, \qquad \text{(A2.4)}$$

where $w$ is a function of $t$ alone, a dot denotes a time derivative and $h$ is a constant, then equations (6) reduce to a single equation

$$\ddot{w} + 6/w^2 = 2hw \qquad \text{(A2.5)}$$

which is Newton's equation for a Kepler harmonic oscillator problem. It admits the first integral

$$\tfrac{1}{2}\dot{w}^2 - 6/w = hw^2 + k \qquad \text{(A2.6)}$$

where $k$ is a constant playing the role of 'energy' and we can explicitly integrate (A2.6). For $k=0$ the results are particularly simple:

$$u=\tfrac{2}{3}\varkappa\coth(\varkappa t)x,\qquad c=\tfrac{1}{3}\varkappa\operatorname{cosech}(\varkappa t)x, \tag{A2.7a, b}$$

$$H=H_0+\tfrac{2}{9}\varkappa^2x^2, \tag{A2.7c}$$

is a solution of (6) where $H_0$ is an arbitrary constant and we have introduced $\varkappa$ in place of $h$ in (A2.4). Another simple solution of (A2.6) is for the case of a flat bottom $h=0$ where we find

$$u=xw^{-3/2}[12(1+Kw)]^{1/2}, \tag{A2.8a}$$

$$c=3^{1/2}xw^{-3/2} \tag{A2.8b}$$

with

$$(12K)^{1/2}Kt \doteq [Kw(1+Kw)]^{1/2}-\sinh^{-1}(Kw)^{1/2} \tag{A2.8c}$$

and $K=6k$ is a constant.

We can also reduce (7) and (8) to ordinary differential equations using scale invariant variables indicated by (A2.2), (A2.3). In this way we could also consider depth functions $H\sim x^{-2}$ and $x^{-1}$ respectively. But the results obtained by this method are too complicated to warrant reproduction here.

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